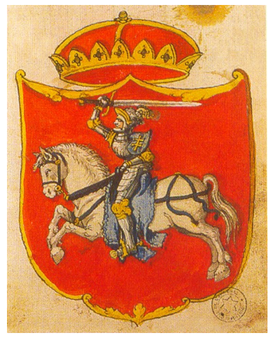
9^{th} Mathematical Contest of Friendship in Honor and Memory of Grand Duchy of Lithuania

24 September 2017



1. The infinite sequence $a_0, a_1, a_2, a_3, \ldots$ is defined by $a_0 = 2$ and
$$a_n = \frac{2a_{n-1}+1}{a_{n-1}+2}, \quad n = 1, 2, 3, \ldots$$
Prove that
$$1 < a_n < 1 + \frac{1}{3^n}$$
for all $n = 1, 2, 3, \ldots$

2. A deck of 52 cards is stacked in a pile facing down. Tom takes the small pile consisting of the seven cards on the top of the deck, turns it around, and places it at the bottom of the deck. All cards are again in one pile, but not all of them face down, since the seven cards at the bottom now face up. Tom repeats this move until all cards face down again. In total, how many moves did Tom make?

3. Let ABC be a triangle with $\angle A = 90°$ and let D be an orthogonal projection of A onto BC. The midpoints of AD and AC are called E and F, respectively. Let M be the circumcentre of $\triangle BEF$.

   Prove that $AC || BM$.

4. Show that there are infinitely many positive integers n such that the number of distinct *odd* prime factors of $n(n+3)$ is a multiple of 3.

   (For instance, $180 = 2^2 \cdot 3^2 \cdot 5$ has two distinct odd prime factors and $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$ has three.)