Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University Problems and Solutions

Paulius Drungilas[1], Artūras Dubickas[2] and Jonas Jankauskas[3]

2015-02-07

PROBLEMS

Problem 1. Let A be the set of all real numbers that can be written as $n - \sqrt[3]{m}$ with some $m, n \in \mathbb{Z}$.

a) Is $\sqrt{2}$ an element of the set A?

b) Is $\sqrt{2}$ a limit point of the set A?

Problem 2. Find the value of the integral

$$\int_{\pi/6}^{\pi/3} \frac{\cos x dx}{\cos x + 2\sin x}.$$

Problem 3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(f(f(x))) + 5f(f(x)) + 2f(x) = 8x$$

for each $x \in \mathbb{R}^+$. (Here, $\mathbb{R}^+$ is the set of all nonnegative real numbers.)

Problem 4. Let n be a positive integer.

a) Prove that there exists a positive integer m divisible by n whose decimal expansion contains only digits from the set $\{0, 1, 8, 9\}$.

b) Prove that such an integer m as in part *a*) can be chosen in the interval $[n, n^4]$.

Each problem is worth 10 points.

[1]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼drungilas/

[2]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼dubickas/

[3]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼jonakank/ and Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

PROBLEMS WITH SOLUTIONS

Problem 1. Let A be the set of all real numbers that can be written as $n - \sqrt[3]{m}$ with some $m, n \in \mathbb{Z}$.

a) Is $\sqrt{2}$ an element of the set A?

b) Is $\sqrt{2}$ a limit point of the set A?

Answer: *a*) no; *b*) yes.

Solution. *a*) Assume that $n - \sqrt[3]{m} = \sqrt{2}$ for some $n, m \in \mathbb{Z}$. If $n = 0$, then $-\sqrt[3]{m} = \sqrt{2}$ yields $m^2 = (-\sqrt[3]{m})^6 = \sqrt{2}^6 = 8$, which is impossible in view of $m \in \mathbb{Z}$. Assume next that $n \neq 0$. Then, by taking the cubes of both sides of $n - \sqrt{2} = \sqrt[3]{m}$, we deduce $n^3 - 3n^2\sqrt{2} + 6n - 2\sqrt{2} = m$, and hence $(n^3 + 6n - m)/(3n^2 + 2) = \sqrt{2}$. Here, the left hand side is a rational number, whereas the left hand side, $\sqrt{2}$, is irrational, a contradiction. This proves that $\sqrt{2} \notin A$.

Alternatively, assuming that $n - \sqrt{2} = \sqrt[3]{m}$ one may observe that the number $n - \sqrt{2}$ is quadratic over the field $\mathbb{Q}$ with conjugate $n + \sqrt{2}$. On the other hand, the number $\sqrt[3]{m}$ is a root of the polynomial $x^3 - m$. It must be quadratic (so not a cubic!) over $\mathbb{Q}$. Hence, the polynomial $x^3 - m$ is reducible over $\mathbb{Q}$. Therefore, is has a rational root, which must be an integer, say t. From $t^3 - m = 0$ we obtain $\sqrt[3]{m} = t \in \mathbb{Q}$, so $\sqrt[3]{m} = n - \sqrt{2}$ is not quadratic over $\mathbb{Q}$. This contradiction proves that $\sqrt{2} \notin A$.

b) Fix some $\alpha \in \mathbb{R}$ and consider the sequence of integers $(a_n)_{n=1}^{\infty}$ defined by $a_n := n^3 - \lfloor 3n^2\alpha \rfloor$. (Here, $\lfloor x \rfloor$ stands for the integral part of a number $x \in \mathbb{R}$.) We claim that $\lim_{n\to\infty}(n - \sqrt[3]{a_n}) = \alpha$. Indeed, using $\lfloor 3n^2\alpha \rfloor = 3n^2\alpha - r_n$, where $r_n \in [0, 1)$, and the identity $(x - y)(x^2 + xy + y^2) = x^3 - y^3$ for $x := n$ and $y := \sqrt[3]{a_n}$, we obtain

$$n - \sqrt[3]{a_n} = \frac{n^3 - a_n}{n^2 + n\sqrt[3]{a_n} + \sqrt[3]{a_n^2}} = \frac{3n^2\alpha - r_n}{n^2 + n\sqrt[3]{a_n} + \sqrt[3]{a_n^2}} = \frac{3\alpha - \frac{r_n}{n^2}}{1 + \frac{\sqrt[3]{a_n}}{n} + \frac{\sqrt[3]{a_n^2}}{n^2}}$$

$$= \frac{3\alpha - \frac{r_n}{n^2}}{1 + \sqrt[3]{1 - \frac{3\alpha}{n} + \frac{r_n}{n^3}} + \sqrt[3]{\left(1 - \frac{3\alpha}{n} + \frac{r_n}{n^3}\right)^2}}$$

for each $n \in \mathbb{N}$. Here, as $n \to \infty$, the nominator tends to 3α and the denominator tends to $1 + 1 + 1 = 3$. Thus, $\lim_{n\to\infty}(n - \sqrt[3]{a_n}) = \alpha$, and hence α is a limit point of the set A. (In particular, we can take $\alpha = \sqrt{2}$.) $\square$

Problem 2. Find the value of the integral

$$\int_{\pi/6}^{\pi/3} \frac{\cos x dx}{\cos x + 2\sin x}.$$

Answer: $\frac{1}{30}\pi + \frac{2}{5}\log(3\sqrt{3} - 4)$.

Solution. Denote the given integral by I and put

$$J := \int_{\pi/6}^{\pi/3} \frac{\sin x dx}{\cos x + 2\sin x}.$$

Then $I + 2J = \int_{\pi/6}^{\pi/3} 1 dx = \pi/3 - \pi/6 = \pi/6$. Setting $f(x) := \log(\cos x + 2\sin x)$ we have

$$f'(x) = \frac{-\sin x + 2\cos x}{\cos x + 2\sin x},$$

so that

$$\begin{aligned} -J + 2I &= \int_{\pi/6}^{\pi/3} f'(x)dx = f(\pi/3) - f(\pi/6) = \log(1/2 + \sqrt{3}) - \log(\sqrt{3}/2 + 1) \\ &= \log(3\sqrt{3} - 4). \end{aligned}$$

Adding this equality multiplied by 2 with $I + 2J = \pi/6$, we derive that

$$5I = (I + 2J) + 2(-J + 2I) = \pi/6 + 2\log(3\sqrt{3} - 4),$$

whence the result. □

Problem 3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(f(f(x))) + 5f(f(x)) + 2f(x) = 8x$$

for each $x \in \mathbb{R}^+$. (Here, $\mathbb{R}^+$ is the set of all nonnegative real numbers.)

Answer: $f(x) = x$.

Solution. It is easy to see that the function $f(x) = x$ satisfies the equation $f(f(f(x))) + 5f(f(x)) + 2f(x) = 8x$. In all what follows, we will prove that this is the only function satisfying the given equation.

Fix an arbitrary number $x \in \mathbb{R}^+$. Put $x_0 := x$ and $x_n := f(x_{n-1})$ for each $n \in \mathbb{N}$. Then, selecting $x = x_n$, we obtain

$$f(x_n) = x_{n+1}, \quad f(f(x_n)) = f(x_{n+1}) = x_{n+2} \quad \text{and} \quad f(f(f(x_n))) = f(x_{n+2}) = x_{n+3},$$

so that $x_n \geqslant 0$ and

$$x_{n+3} + 5x_{n+2} + 2x_{n+1} - 8x_n = 0$$

for each integer $n \geqslant 0$. Since the characteristic equation of this linear recurrence relation is

$$\lambda^3 + 5\lambda^2 + 2\lambda - 8 = (\lambda - 1)(\lambda + 2)(\lambda + 4) = 0,$$

we must have

$$x_n = A + B(-2)^n + C(-4)^n$$

for some real numbers A, B, C and each integer $n \geqslant 0$.

If at least one of the numbers B, C is nonzero, then $x_n < 0$ for some $n > 0$, which is impossible. (Indeed, if $C \neq 0$, then one can take a large even n if $C < 0$ and a large odd n if $C > 0$. Similarly, if $C = 0$, but $B \neq 0$, then one gets a negative x_n, by taking a large even n if $B < 0$ and a large odd n if $B > 0$.) Thus, $B = C = 0$ and $x_n = A$ is the only solution of the linear recurrence relation. In particular, substituting $n = 0$ and $n = 1$ into $x_n = A$ we find that $x = x_0 = A = x_1 = f(x_0) = f(x)$. Therefore, for each $x \in \mathbb{R}^+$ the only possible value for $f(x)$ is x, which implies that the only possible function is $f(x) = x$. □

Problem 4. Let n be a positive integer.

a) Prove that there exists a positive integer m divisible by n whose decimal expansion contains only digits from the set $\{0, 1, 8, 9\}$.

b) Prove that such an integer m as in part *a*) can be chosen in the interval $[n, n^4]$.

Solution. For $n = 1$ we can take $m = 1$, so in the sequel we will assume that $n \geqslant 2$.

a) Write $n = 2^a 5^b N$, where $a, b \geqslant 0$ and $\gcd(N, 10) = 1$. By Euler's theorem, $10^{\varphi(N)} - 1$ is divisible by N, where $\varphi(N)$ is Euler's totient function. Setting $c := \max(a, b)$ we see that 10^c is divisible by $2^a 5^b$. Hence, $n = 2^a 5^b N$ divides the integer

$$10^c(10^{\varphi(N)} - 1) = 10^{c+\varphi(N)} - 10^c$$

consisting of $\varphi(N)$ digits 9 and c digits 0.

Here is an alternative solution: consider the integers of the form

$$0, 1, 11, 111, 1111, 11111, \ldots.$$

At least two of them, say $a > b$, give the same remainder modulo n. Their difference $m = a - b$ is thus divisible by n and contains the digits 1 and 0 only.

Note that if in the second solution we consider the first $n+1$ integers (up to the integer $\underbrace{1\ldots1}_{n \text{ times}}$), then we obtain the bound $m \leqslant \frac{10^n-1}{9}$, which is too large to solve part *b*), but gives an idea how this can be done.

b) Take the smallest integer k satisfying $n < 2^k$, namely, $k := \lfloor \log_2 n \rfloor + 1$, where $\lfloor x \rfloor$ stands for the integral part of a number $x \in \mathbb{R}$. Consider the set M consisting of 2^k nonnegative integers whose k decimal digits all belong to the set $\{0, 1\}$. Since $n < 2^k$, at least two elements of M, say $a > b$, give the same remainder modulo n. Thus, their difference $a - b$ is divisible by n. Clearly, as the digits of a and b are in the set $\{0, 1\}$, the digits of the positive integer $a - b$ are all in the set $\{0, 1, 8, 9\}$. Furthermore, since $k = \lfloor \log_2 n \rfloor + 1 \geqslant 1$ and the difference $a - b$ does not exceed the largest element of M, we derive that

$$a - b \leqslant \underbrace{1 \ldots 1}_{k \text{ times}} = \frac{10^k - 1}{9} < \frac{10^k}{9} = 10^{\lfloor \log_2 n \rfloor} \cdot \frac{10}{9} < 10^{\lfloor \log_2 n \rfloor} \cdot 1.6^{\lfloor \log_2 n \rfloor}$$

$$= 16^{\lfloor \log_2 n \rfloor} \leqslant 16^{\log_2 n} = 2^{4 \log_2 n} = n^4.$$

Therefore, we can take $m = a - b \in [n, n^4]$ for each $n \geqslant 2$. $\square$