Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University Problems and Solutions

Paulius Drungilas[1], **Artūras Dubickas**[2]

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PROBLEMS

Problem 1. Let R be the region consisting of all the points (x, y) in the Cartesian plane $\mathbb{R}^2$ satisfying two inequalities $|x| - y^2 \leqslant 1$ and $|y| \leqslant 1$. Find the area of R.

Problem 2. Find all prime numbers in the infinite sequence of integers

$$101,\ 10101,\ 1010101,\ 101010101, \dots.$$

Problem 3. Let $n \geqslant 2$ be a fixed positive integer. Find all positive integers k with the property that the kth derivative $f^{(k)}(x)$ of an arbitrary infinitely many times differentiable function $f : \mathbb{R} \to \mathbb{R}$ with at least n distinct zeros in the interval $[0, 1]$ (namely, $f(a_1) = f(a_2) = \dots = f(a_n) = 0$ for some $a_1 < a_2 < \dots < a_n \in [0, 1]$) has at least one zero in the interval $[0, 1]$.

Problem 4. Find all pairs of positive integers (m, n) satisfying

$$m^{n^m} = n^{m^n}.$$

Each problem is worth 10 points.

[1]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼drungilas/

[2]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼dubickas/

PROBLEMS WITH SOLUTIONS

Problem 1. Let R be the region consisting of all the points (x, y) in the Cartesian plane $\mathbb{R}^2$ satisfying two inequalities $|x| - y^2 \leqslant 1$ and $|y| \leqslant 1$. Find the area of R.

Answer: 16/3.

Solution. Note that a point $(x, y) \in \mathbb{R}^2$ belongs to the region R if and only if the point $(x, -y)$ belongs to R. Hence the region R is symmetric with respect to the axis Ox. Denote by R_0 the region consisting of the points of R in the halfplane $x \geqslant 0$. Then, by symmetry, the area of R is twice the area of R_0.

Evidently, the region R_0 consists of the points $(x, y) \in \mathbb{R}^2$ satisfying $0 \leqslant x \leqslant y^2 + 1$ and $-1 \leqslant y \leqslant 1$. For easier integration we can interchange x and y and get the system of inequalities $0 \leqslant y \leqslant x^2 + 1$ and $-1 \leqslant x \leqslant 1$. It follows that the area of R_0 equals

$$\int_{-1}^{1} (x^2 + 1)\, dx = \left(\frac{x^3}{3} + x\right)\Bigg|_{-1}^{1} = \frac{8}{3}.$$

Hence, the area of R is equal to $2 \cdot \frac{8}{3} = \frac{16}{3}$. □

Problem 2. Find all prime numbers in the infinite sequence of integers

$$101,\ 10101,\ 1010101,\ 101010101, \ldots .$$

Answer: The number 101 is the only one prime number of this sequence.

Solution. One can easily check that the number 101 is prime. (Indeed, otherwise it should be divisible by a prime less than $\sqrt{101} < 11$, so by $2, 3, 5$ or 7. Evidently, 101 is not divisible by any of these numbers.)

Now, we will show that all the other elements of this sequence are composite. The element of the sequence $N = 101\ldots01$ containing k digits 1 can be written as

$$N = 1 + 100 + 100^2 + \cdots + 100^{k-1} = \frac{100^k - 1}{100 - 1} = \frac{10^{2k} - 1}{99} = \frac{(10^k - 1)(10^k + 1)}{9 \cdot 11}.$$

Here we have $k \geqslant 3$, so the integers $a := (10^k - 1)/9$ and $b := 10^k + 1$ are both greater than 11. Furthermore, the prime number 11 must divide a or b. So $N = ab/11$ is the product of two integers greater than 1: either $a/11$ and b or a and $b/11$. Therefore, N is a composite integer. □

Problem 3. Let $n \geqslant 2$ be a fixed positive integer. Find all positive integers k with the property that the kth derivative $f^{(k)}(x)$ of an arbitrary infinitely many times differentiable function $f : \mathbb{R} \to \mathbb{R}$ with at least n distinct zeros in the interval $[0, 1]$ (namely, $f(a_1) = f(a_2) = \cdots = f(a_n) = 0$ for some $a_1 < a_2 < \cdots < a_n \in [0, 1]$) has at least one zero in the interval $[0, 1]$.

Answer: $k \in \{1, 2, \ldots, n-1\}$.

Solution. Let $\mathcal{K}$ be the set of all positive integers k with the required property.

Note that, by Rolle's theorem, for each $i = 1, \ldots, n-1$ there exist $b_i \in (a_i, a_{i+1})$ such that $f'(b_i) = 0$. In particular, $f'(x)$ is vanishing at at $b_1 \in (0,1)$, and so $k = 1 \in \mathcal{K}$. For $n \geqslant 3$, applying the same argument to the function $f'(x)$ we find that for each $i = 1, \ldots, n-2$ there are $c_i \in (b_i, b_{i+1})$ satisfying $f''(c_i) = 0$, so that $k = 2 \in \mathcal{K}$, etc. This argument can be applied $n-1$ times up to the derivative $f^{(n-1)}(x)$ having at least one point $d_1 \in (0,1)$ such that $f^{(n-1)}(d_1) = 0$. Consequently, $\mathcal{K}$ contains all the integers k in the range $1 \leqslant k \leqslant n-1$.

Suppose now that $k \geqslant n$. We will prove that $k \notin \mathcal{K}$.

First proof. Fix any $a_1 < a_2 < \cdots < a_n \in [0,1]$ and consider the function

$$f(x) := p(x)e^{\delta x}, \quad \text{where} \quad p(x) := (x-a_1)(x-a_2)\ldots(x-a_n)$$

and $\delta > 0$ is a sufficiently small constant to be specified later. We claim that for this function $f(x)$ (evidently, it is infinitely many times differentiable function with n distinct zeros in $[0,1]$) the inequality $f^{(k)}(x) > 0$ holds for each $x \in [0,1]$, so that the kth derivative $f^{(k)}(x)$ has no zeros in the interval $[0,1]$.

In order to prove this inequality we first observe that $p^{(N)}(x) \equiv 0$ for each integer $N > n$. Hence, in view of $k \geqslant n$ and $p^{(n)}(x) = n!$ we get

$$\begin{aligned} f^{(k)}(x) &= (p(x)e^{\delta x})^{(k)} = \sum_{j=0}^{k} \binom{k}{j} p^{(j)}(x)(e^{\delta x})^{(k-j)} = \sum_{j=0}^{n} \binom{k}{j} p^{(j)}(x)\delta^{k-j}e^{\delta x} \\ &= g_k(x)\delta^{k-n}e^{\delta x}, \end{aligned}$$

where

$$g_k(x) := \binom{k}{n} n! + \binom{k}{n-1} p^{(n-1)}(x)\delta + \cdots + \binom{k}{0} p(x)\delta^n.$$

Now, set

$$M := \max_{0 \leqslant j \leqslant n-1} \max_{0 \leqslant x \leqslant 1} |p^{(j)}(x)|$$

and choose positive $\delta < \min(1, 1/(2Mn^{n+1}))$. Then, using $k - n + 1 \geqslant k/n$ we find that

$$\binom{k}{n} n! = k(k-1)\ldots(k-n+1) \geqslant \left(\frac{k}{n}\right)^n$$

and

$$\left| \binom{k}{n-1} p^{(n-1)}(x)\delta + \cdots + \binom{k}{0} p(x)\delta^n \right| < nk^{n-1}M\frac{1}{2Mn^{n+1}} = \frac{k^{n-1}}{2n^n}.$$

Thus, for each $x \in [0,1]$ we derive that

$$g_k(x) > \left(\frac{k}{n}\right)^n - \frac{k^{n-1}}{2n^n} \geqslant \frac{k^n}{2n^n} > 0.$$

Consequently, $f^{(k)}(x) = g_k(x)\delta^{k-n}e^{\delta x} > 0$ for every $x \in [0, 1]$, as claimed.

Second proof. Alternatively, for each $k \geqslant n$ one can construct a function f depending on k. For example, take any k distinct real numbers $b_1, \dots, b_k$ in the interval $[0, 1]$. Evidently, the polynomial $f(x) := (x - b_1) \dots (x - b_k)$ is infinitely many times differentiable function $\mathbb{R} \to \mathbb{R}$ with at least n distinct zeros in $[0, 1]$ (in fact, f has $k \geqslant n$ distinct zeros in $[0, 1]$). Moreover, its kth derivative $f^{(k)}(x) = k!$ has no zeros in $[0, 1]$. □

Problem 4. Find all pairs of positive integers (m, n) satisfying

$$m^{n^m} = n^{m^n}.$$

Answer: $(m, n) = (t, t)$ with $t \in \mathbb{N}$.

Solution. Obviously, each pair $(m, n) = (t, t)$, where $t \in \mathbb{N}$, satisfies the given equation. Below, we will show that no pair (m, n), where $m \neq n$, can satisfy this equation. Without restriction of generality we may assume that $m > n$.

Clearly, the equality cannot hold for $n = 1$ and $m > 1$. If $n \geqslant 3$ then $n^m > m^n$, since $m/\log m > n/\log n$. (Indeed, the derivative of the function $f(x) = x/\log x$ is $f'(x) = (\log x - 1)/(\log x)^2$, so $f(x)$ is increasing in the interval $[3, +\infty) \subset (e, \infty)$.) Combining this with $m > n$ we obtain $m^{n^m} > n^{m^n}$.

Finally, in case $n = 2$, m cannot be equal to 3 (since positive powers of 2 and 3 cannot be equal), so we must have $m \geqslant 4$. Then, by induction on m, using the inequalities $2^4 \geqslant 4^2$ (for the basis of induction) and $2 \geqslant 25/16 \geqslant ((m + 1)/m)^2$ (for the induction step $m \to m + 1$), we derive that $2^m \geqslant m^2$ for $m \geqslant 4$. Hence, for $m \geqslant 4$ we have $m^{2^m} > 2^{m^2}$. □