Mathematical Competition for Students (MIFMO)
of the Department of Mathematics and Informatics
of Vilnius University

2019-02-09

(organized by Paulius Drungilas and Artūras Dubickas)

Problem 1. Show that there are infinitely many pairs of irrational numbers (a, b) such that

$$\frac{2a + 3b + 2019(ab)^3}{(ab + 1)^2} \in \mathbb{Q}.$$

Problem 2. Prove that there exist 3×3 matrices A and B such that $ABAB = 0$ and $BABA \neq 0$.

Problem 3. Let S be the set $\{2^k : k \in \mathbb{Z}\}$ and let $f : [1, \infty) \to (0, \infty)$ be a continuous function with the following property: for each $a \in S$ the equation $f(x) = ax^2$ has a solution in $x \in [1, \infty)$.

a) Prove that for each $a > 0$ there exist infinitely many $x > 1$ satisfying $f(x) = ax^2$.

b) Is there a function f as described above which is increasing in $[1, \infty)$?

Problem 4. A group of Facebook contains 2019 members. Some of them are friends. Anybody can send a message to any of its friends. Suppose Angela and Theresa have the largest and the smallest number of friends respectively (any other member of the group has less friends than Angela and more friends than Theresa). Let N be the sum of the friends of Angela and Theresa (joint friends are counted twice). Donald and Vladimir are also members of the group, but they are not friends. Find the smallest possible N such that Donald can always send a message to Vladimir so that at most 3 other members of the group will be able to read it during its transfer.

Each problem is worth 10 points.