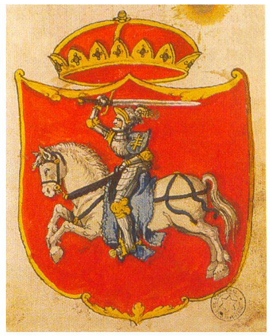
11^{th} Mathematical Contest of Friendship in Honor and Memory of Grand Duchy of Lithuania

29 September 2019



1. Let x, y, z be positive numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that
$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geqslant \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

2. Every cell of a 20×20 table has to be coloured black or white (there are 2^{400} such colourings in total). Given any colouring P, we consider division of the table into rectangles with sides in the grid lines where no rectangle contains more than two black cells and where the number of rectangles containing at most one black cell is the least possible. We denote this smallest possible number of rectangles containing at most one black cell by $f(P)$. Determine the maximum value of $f(P)$ as P ranges over all colourings.

3. Let ABC be an acute triangle with orthocenter H and circumcenter O. The perpendicular bisector of segment CH intersects the sides AC and BC in points X and Y, respectively. The lines XO and YO intersect the side AB in points P and Q, respectively. Prove that if $XP + YQ = AB + XY$ then $\angle OHC = 90^\circ$.

4. Determine all pairs of prime numbers (p, q) such that $p^2 + 5pq + 4q^2$ is a square of an integer.