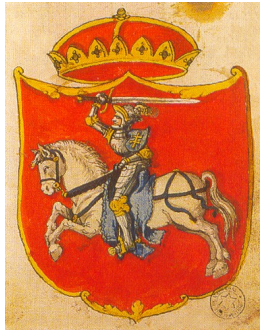


**13<sup>th</sup> Mathematical Contest of Friendship  
in Honor and Memory of Grand Duchy of Lithuania**

**3 October 2021**



1. Prove that for any polynomial  $f(x)$  (with real coefficients) there exist polynomials  $g(x)$  and  $h(x)$  (with real coefficients) such that

$$f(x) = g(h(x)) - h(g(x)).$$

2. Every number in the sequence  $1, 2, \dots, 2021$  is either white or black. At one step Alice can choose three numbers of the sequence and change the colour of each of them (white to black and black to white) if one of those three numbers is the arithmetic mean of the other two. Alice wants to perform several steps so that at the end all the numbers in the sequence are black. For which initial colourings of numbers can Alice achieve this?
3. Let  $ABCD$  be a convex quadrilateral satisfying

$$\angle ADB + \angle ACB = \angle CAB + \angle DBA = 30^\circ, \quad AD = BC.$$

Prove that there exists a right-angled triangle with side lengths  $AC$ ,  $BD$ ,  $CD$ .

4. A triplet of positive integers  $(x, y, z)$  satisfying  $x, y, z > 1$  and

$$x^3 - yz^3 = 2021$$

is called *primary* if at least two of the integers  $x, y, z$  are prime numbers.

- a) Find at least one primary triplet.  
b) Show that there are infinitely many primary triplets.