

**Mathematical Competition for Students (MIFMO)**  
**of the Department of Mathematics and Informatics**  
**of Vilnius University**

2022-02-12

(organized by Paulius Drungilas and Artūras Dubickas)

**Problem 1.**

- a) Prove that there exist infinitely many integers  $n$  such that  $n$ ,  $n + 1$  and  $n + 2$  are each the sum of two squares of integers.
- b) Does there exist an integer  $n$  such that each of the numbers  $n$ ,  $n + 1$ ,  $n + 2$  and  $n + 3$  is the sum of two squares of integers?

**Problem 2.** Find the value of

$$\int_0^\pi \cos^{2022}(x) \cos(100x) dx.$$

**Problem 3.** Let  $f : (\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}$  be a function satisfying

$$f(x, x) = f(y, y)$$

for all  $x, y \in \mathbb{Z}$  and

$$f(x, f(y, z)) = f(x, y) + z$$

for all  $x, y, z \in \mathbb{Z}$ . (Here,  $\mathbb{Z}$  denotes the set of all integers.)

Find all possible values of  $f(1000, 2022)$ .

**Problem 4.** A subset of a group of students is called an *ideal company* if

- (i) it contains at least one girl and at least one boy;
- (ii) each boy of this subset likes every girl of this subset;
- (iii) nobody can be added to this subset without violation of rule (ii).

Find the maximal number of ideal companies in a group of 10 girls and 20 boys.

**Each problem is worth 10 points.**