Mathematical Competition for Students (MIFMO)
of the Department of Mathematics and Informatics
of Vilnius University

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(organized by Paulius Drungilas and Artūras Dubickas)

Problem 1. Let m and n be two positive integers. Prove that each function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(-x)$ and $f(x) + f(m - x) = n$ for every $x \in \mathbb{R}$ is periodic and give an example of such a function which is continuous and not a constant.

Problem 2. You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, 3, \ldots$. Each ball has probability $\frac{1}{2^n}$ of being put into box labelled n. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

Problem 3. Consider all possible four term sequences of real numbers x_1, x_2, x_3, x_4 satisfying the following three conditions:

(i) $x_3 = x_1 + x_2$;
(ii) $x_4 = x_3 + x_2$;
(iii) there exist real numbers a, b, c such that $\cos(x_i) = ai^2 + bi + c$ for $i = 1, 2, 3, 4$.

Determine the maximal value of

$$\cos(x_1) - \cos(x_4).$$

Problem 4. For any integers a and b let $S(a, b)$ be the infinite set of integers of the form $n^2 + an + b$, where n runs through all integers. Find the largest positive integer m for which there exist m pairs of positive integers (a_i, b_i), $i = 1, \ldots, m$, such that the m sets $S(a_i, b_i)$, $i = 1, \ldots, m$, are pairwise disjoint, namely, $S(a_i, b_i) \cap S(a_j, b_j) = \emptyset$ whenever $1 \leqslant i < j \leqslant m$.

Each problem is worth 10 points.