# Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2023

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## PROBLEMS

**Problem 1.** Let $m$ and $n$ be two positive integers. Prove that each function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(-x)$ and $f(x) + f(m - x) = n$ for every $x \in \mathbb{R}$ is periodic and give an example of such a function which is continuous and not a constant.

**Problem 2.** You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, 3, \ldots$. Each ball has probability $\frac{1}{2^n}$ of being put into box labelled $n$. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

**Problem 3.** Consider all possible four term sequences of real numbers $x_1, x_2, x_3, x_4$ satisfying the following three conditions:

(i) $x_3 = x_1 + x_2$;

(ii) $x_4 = x_3 + x_2$;

(iii) there exist real numbers $a, b, c$ such that $\cos(x_i) = ai^2 + bi + c$ for $i = 1, 2, 3, 4$.

Determine the maximal value of

$$\cos(x_1) - \cos(x_4).$$

**Problem 4.** For any integers $a$ and $b$ let $S(a, b)$ be the infinite set of integers of the form $n^2 + an + b$, where $n$ runs through all integers. Find the largest positive integer $m$ for which there exist $m$ pairs of positive integers $(a_i, b_i)$, $i = 1, \ldots, m$, such that the $m$ sets $S(a_i, b_i)$, $i = 1, \ldots, m$, are pairwise disjoint, namely, $S(a_i, b_i) \cap S(a_j, b_j) = \emptyset$ whenever $1 \leqslant i < j \leqslant m$.

**Each problem is worth 10 points.**

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## PROBLEMS WITH SOLUTIONS

**Problem 1.** Let $m$ and $n$ be two positive integers. Prove that each function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(-x)$ and $f(x) + f(m - x) = n$ for every $x \in \mathbb{R}$ is periodic and give an example of such a function which is continuous and not a constant.

*Answer:* For example, $f(x) = \frac{n}{2} + \cos\left(\frac{\pi x}{m}\right)$.

*Solution.* Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying both conditions. Then, in view of $f(m - x) = f(x - m)$ we obtain

$$n = f(x) + f(m - x) = f(x) + f(x - m).$$

Inserting $x + m$ in place of $x$ we find that

$$n = f(x + m) + f(x + m - m) = f(x + m) + f(x) = f(x) + f(x + m).$$

Hence, $f(x+m) = f(x-m)$ for each $x \in \mathbb{R}$, which implies that $f$ is periodic with period $2m$.

As an example we can take, e. g., $f(x) = \frac{n}{2} + \cos\left(\frac{\pi x}{m}\right)$. Then, $f$ is continuous, not a constant, and for each $x \in \mathbb{R}$ satisfies

$$f(x) + f(m - x) = n + \cos\left(\frac{\pi x}{m}\right) + \cos\left(\frac{\pi(m - x)}{m}\right) = n + \cos\left(\frac{\pi x}{m}\right) - \cos\left(\frac{\pi x}{m}\right) = n$$

and $f(x) = f(-x)$. □

**Problem 2.** You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, 3, \ldots$. Each ball has probability $\frac{1}{2^n}$ of being put into box labelled $n$. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

*Answer:* $\frac{5}{7}$.

*Solution.* Notice that the answer is the sum of the probabilities that boxes $1, 2, 3, \ldots$, respectively, contain at least 2 balls, since those events are mutually exclusive. For box $n$, the probability $p_n$ of having at least 2 balls in it equals

$$p_n = \left(\frac{1}{2^n}\right)^3 + 3\left(\frac{1}{2^n}\right)^2\left(1 - \frac{1}{2^n}\right) = \frac{3}{4^n} - \frac{2}{8^n}.$$

Summing two geometric progressions over $n = 1, 2, 3, \ldots$ we obtain

$$p = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty}\left(\frac{3}{4^n} - \frac{2}{8^n}\right) = \sum_{n=1}^{\infty}\frac{3}{4^n} - \sum_{n=1}^{\infty}\frac{2}{8^n} = 1 - \frac{2}{7} = \frac{5}{7},$$

so the probability that some box will contain at least 2 balls is equal to $\frac{5}{7}$. □

**Problem 3.** Consider all possible four term sequences of real numbers $x_1, x_2, x_3, x_4$ satisfying the following three conditions:

(i) $x_3 = x_1 + x_2$;
(ii) $x_4 = x_3 + x_2$;
(iii) there exist real numbers $a, b, c$ such that $\cos(x_i) = ai^2 + bi + c$ for $i = 1, 2, 3, 4$.

Determine the maximal value of

$$\cos(x_1) - \cos(x_4).$$

*Answer:* $3\sqrt{13} - 9$.

*Solution.* We first show that condition (iii) is equivalent to

(iv) $\cos(x_4) = \cos(x_1) + 3\cos(x_3) - 3\cos(x_2)$.

Indeed, (iii) means that $\cos(x_1) = a + b + c$, $\cos(x_2) = 4a + 2b + c$, $\cos(x_3) = 9a + 3b + c$ and $\cos(x_4) = 16a + 4b + c$ for some $a, b, c \in \mathbb{R}$. Thus, $\cos(x_3) - \cos(x_2) = 5a + b$ and $\cos(x_4) - \cos(x_1) = 15a + 3b = 3(5a + b) = 3\cos(x_3) - 3\cos(x_2)$, which implies (iv). On the other hand, assume that $x_1, x_2, x_3, x_4$ satisfy (iv). Note that we can always choose $a, b, c \in \mathbb{R}$ satisfying

$$\begin{cases} a + b + c = \cos(x_1), \\ 4a + 2b + c = \cos(x_2), \\ 9a + 3b + c = \cos(x_3), \end{cases}$$

since the corresponding $3 \times 3$ determinant of the linear system is nonzero. Then, using (iv) and the previous linear system we obtain

$$\cos(x_4) = (a + b + c) + 3(9a + 3b + c) - 3(4a + 2b + c) = 16a + 4b + c,$$

which implies (iii). Therefore, conditions (iii) and (iv) are equivalent.

Now, applying (iv) we deduce

$$\cos(x_4) - \cos(x_1) = 3(\cos(x_3) - \cos(x_2)) = -6\sin\left(\frac{x_3 + x_2}{2}\right)\sin\left(\frac{x_3 - x_2}{2}\right),$$

which equals $-6\sin\left(\frac{x_4}{2}\right)\sin\left(\frac{x_1}{2}\right)$ by (i) and (ii). Setting $x = \sin\left(\frac{x_4}{2}\right)$ and $y = \sin\left(\frac{x_1}{2}\right)$ we can rearrange this equality into $(1 - 2x^2) - (1 - 2y^2) = -6xy$, which is equivalent to $x^2 - y^2 = 3xy$. Hence, $y = x\frac{-3\pm\sqrt{13}}{2}$. We are maximizing the expression

$$\cos(x_1) - \cos(x_4) = (1 - 2y^2) - (1 - 2x^2) = 2(x^2 - y^2) = 6xy,$$

so $x$ and $y$ must be of the same sign, which means that $y = x\frac{-3+\sqrt{13}}{2}$. Thus, $|y| \leqslant |x|$. Also, $x = \sin\left(\frac{x_4}{2}\right)$ can attain any value in the interval $[-1, 1]$. Thus, to maximize $6xy$ we can simply set $x = 1$ and $y = \frac{-3+\sqrt{13}}{2}$, which gives the maximal value of $6xy$ equal to $3(-3 + \sqrt{13}) = 3\sqrt{13} - 9$. □

**Problem 4.** For any integers $a$ and $b$ let $S(a,b)$ be the infinite set of integers of the form $n^2+an+b$, where $n$ runs through all integers. Find the largest positive integer $m$ for which there exist $m$ pairs of positive integers $(a_i,b_i)$, $i=1,\dots,m$, such that the $m$ sets $S(a_i,b_i)$, $i=1,\dots,m$, are pairwise disjoint, namely, $S(a_i,b_i)\cap S(a_j,b_j)=\emptyset$ whenever $1\leqslant i<j\leqslant m$.

*Answer:* $m=2$.

*Solution.* The two sets $S(0,0)$ and $S(0,2)$ are disjoint. Indeed, otherwise we must have $n_1^2=n_2^2+2$ for some $n_1,n_2\in\mathbb{Z}$, which is clearly impossible. (For example, without loss of generality we may assume that $n_1>n_2\geqslant 0$. Then, $n_2\neq 0$ and from $n_1\geqslant n_2+1$ we deduce $n_1^2-n_2^2\geqslant (n_2+1)^2-n_2^2=2n_2+1\geqslant 3>2$.)

In all what follows we will show that for any three pairs $(a_1,b_1)$, $(a_2,b_2)$, $(a_3,b_3)\in\mathbb{Z}^2$ the sets $S(a_1,b_1)$, $S(a_2,b_2)$, $S(a_3,b_3)$ cannot be pairwise disjoint, so the largest positive integer $m$ is equal to 2. Observe that if $a$ is even, then

$$n^2+an+b=\left(n+\frac{a}{2}\right)^2+b-\frac{a^2}{4},$$

so $S(a,b)=S(0,b')$, where $b'=b-a^2/4\in\mathbb{Z}$. Likewise, if $a$ is odd, then $a-1$ is even and so

$$n^2+an+b=\left(n+\frac{a-1}{2}\right)^2+n+b-\frac{(a-1)^2}{4},$$

giving $S(a,b)=S(1,b')$ for $b'=b-(a-1)^2/4\in\mathbb{Z}$.

Suppose that among $a_1,a_2,a_3$ there is an even integer and an odd integer. Without restriction of generality we may assume that $a_1$ is even and $a_2$ is odd. Then, the sets $S(a_1,b_1)=S(0,b_1')$ and $S(a_2,b_2)=S(1,b_2')$ are not disjoint, because

$$n^2+b_1'=n^2+n+b_2'$$

for $n=b_1'-b_2'\in\mathbb{Z}$. Consequently, three (not necessarily distinct) integers $a_1,a_2,a_3$ must be either all even or all odd.

If they are all even, then the three sets $S(a_i,b_i)=S(0,b_i')$, $i=1,2,3$, cannot be disjoint. Indeed, among three (not necessarily distinct) integers $b_1',b_2',b_3'$, either at least two have different parity or they are all three of the same parity. In the first case, without restriction of generality we may assume that $b_1'-b_2'=2l+1$ with $l\in\mathbb{Z}$. Then,

$$l^2+b_1'=(l+1)^2+b_2',$$

which shows that the intersection $S(0,b_1')\cap S(0,b_2')$ is not empty. In the alternative case, when all three integers $b_1',b_2',b_3'$ have the same parity, at least two of them are equal modulo 4. Then, without restriction of generality we may assume that $b_1'-b_2'=4l$ with $l\in\mathbb{Z}$. This leads to

$$(l-1)^2+b_1'=(l+1)^2+b_2',$$

and shows that the intersection $S(0, b'_1) \cap S(0, b'_2)$ is not empty.

It remains to consider the case when the integers $a_1, a_2, a_3$ are all odd. This time, we need to show that the three sets $S(a_i, b_i) = S(1, b'_i)$, $i = 1, 2, 3$, cannot be disjoint. Note that among three (not necessarily distinct) integers $b'_1, b'_2, b'_3$ at least two have the same parity. Then, without restriction of generality we may assume that $b'_1 - b'_2 = 2l$ with $l \in \mathbb{Z}$. From the identity

$$(l-1)^2 + l - 1 + b'_1 = l^2 + l + b'_2$$

it follows that the intersection $S(1, b'_1) \cap S(1, b'_2)$ is not empty, which finishes the proof. $\square$