Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2024

Paulius Drungilas[1], **Artūras Dubickas**[2]

2024-02-10

PROBLEMS

Problem 1. As a monkey was carrying three coconuts to the top of a multistory building, one of the nuts fell to the ground from the 16th floor and crashed. The monkey decided to determine the lowest floor (from the first to 16th) for the nut to crash when dropped to the ground. The monkey still has two coconuts. Any of those two can be dropped from any of the floors, and then picked up and used again for a new trial if it did not crash. Is it true that at most five trials are enough for the monkey to determine the lowest floor from which a coconut crashes?

Problem 2. Let $n > m$ be positive integers. Suppose that a complex number z_0, with $|z_0| = 1$, is a root of the polynomial $z^n - z^m + 1$. Prove that z_0 is a root of unity, i.e., there exists a positive integer N such that $z_0^N = 1$.

Problem 3. Find all functions $f : (0, +\infty) \to (0, +\infty)$ satisfying

$$f^2(x) \geqslant f(x+y)(f(x)+y)$$

for all $x, y > 0$.

Problem 4. Do there exist positive integers $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{101}$ such that

$$a_1 a_2 \ldots a_{101} = \sum_{1 \leqslant i < j \leqslant 101} \operatorname{lcm}(a_i, a_j),$$

where $\operatorname{lcm}(a, b)$ stands for the least common multiple of positive integers a and b?

Each problem is worth 10 points.

[1]Institute of Mathematics, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania, `http://www.mif.vu.lt/~drungilas/`

[2]Institute of Mathematics, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania, `http://www.mif.vu.lt/~dubickas/`

PROBLEMS WITH SOLUTIONS

Problem 1. As a monkey was carrying three coconuts to the top of a multistory building, one of the nuts fell to the ground from the 16th floor and crashed. The monkey decided to determine the lowest floor (from the first to 16th) for the nut to crash when dropped to the ground. The monkey still has two coconuts. Any of those two can be dropped from any of the floors, and then picked up and used again for a new trial if it did not crash. Is it true that at most five trials are enough for the monkey to determine the lowest floor from which a coconut crashes?

Answer: Yes.

Solution. We will present a strategy showing that five trials are enough. The monkey starts the first trial from the 5th floor with one of the nuts. If the coconut did crash, then with the only remaining nut it does consequently from the first, the second, the third and the fourth floor (if necessary) until the second nut will crash. If it will not, then the lowest floor is 5th, so at most five trials are enough in that case. Assume that after the first trial (from the 5th floor) the coconut is unbroken. Then, the second trial is to drop the same nut from the 9th floor. If the nut is crashed, then it drops the second nut from the 6th, 7th, and then 8th floor (if necessary). Again at most five trials are enough. If after the second trial (from the 9th floor) the nut is still unbroken, then the third trial is to drop it from the 12th floor. If crashed, then the monkey uses the second nut from the 10th and 11th floors (if necessary), so five trials are enough. If unbroken, then the fourth trial is to drop the same nut from the 14th floor. Then, in the final fifth trial it drops the second nut from the 13th floor (if the previous crashed) or from the 15th floor (if not). □

Problem 2. Let $n > m$ be positive integers. Suppose that a complex number z_0, with $|z_0| = 1$, is a root of the polynomial $z^n - z^m + 1$. Prove that z_0 is a root of unity, i.e., there exists a positive integer N such that $z_0^N = 1$.

Solution. Assume that a complex number z_0, with $|z_0| = 1$, is a root of the polynomial $z^n - z^m + 1$. Then, $z_0^n - z_0^m + 1 = 0$. Setting $X = -z_0^n$ and $Y = z_0^m$ we obtain $|X| = |Y| = 1$ and $X + Y = 1$. Note that, for any complex number a satisfying $|a| = 1$, we have $1 = |a|^2 = a\overline{a}$, and hence $\overline{a} = a^{-1}$. Therefore,

$$1 = X + Y = \overline{X + Y} = \overline{X} + \overline{Y} = X^{-1} + Y^{-1} = \frac{X + Y}{XY} = \frac{1}{XY} = \frac{1}{-z_0^{m+n}}.$$

By squaring this equality we get $z_0^{2m+2n} = 1$, which gives the desired conclusion with $N = 2m + 2n$. □

Problem 3. Find all functions $f : (0, +\infty) \to (0, +\infty)$ satisfying

$$f^2(x) \geqslant f(x+y)(f(x)+y)$$

for all $x, y > 0$.

Answer: There are no such functions.

Solution. Assume that f is such a function. Then, for any $x, y > 0$ we have

$$f(x) - f(x+y) \geqslant f(x) - \frac{f^2(x)}{f(x)+y} = \frac{yf(x)}{f(x)+y} = \frac{1}{\frac{1}{y} + \frac{1}{f(x)}} > 0,$$

so f is decreasing. Fix $x > 0$ and choose $n \in \mathbb{N}$ so large that $nf(x+1) \geqslant 1$. Inserting into the above formula $x+k/n$, with k in the range $0 \leqslant k < n$, in place of x and $y = 1/n$, and using $f(x+k/n) > f(x+1)$ we find that

$$f(x+k/n) - f(x+(k+1)/n) \geqslant \frac{1}{n + \frac{1}{f(x+k/n)}} > \frac{1}{n + \frac{1}{f(x+1)}} \geqslant \frac{1}{n+n} = \frac{1}{2n}.$$

Adding n such formulas for $k = 0, 1, \ldots, n-1$ gives $f(x) - f(x+1) \geqslant 1/2$. Fix an integer $m \geqslant f(x)$. Adding $2m$ formulas $f(x+j) - f(x+j+1) \geqslant 1/2$ for $j = 0, 1, \ldots, 2m-1$ we derive $f(x) - f(x+2m) \geqslant 2m \cdot (1/2) = m$, so $f(x+2m) \leqslant f(x) - m \leqslant 0$, which is impossible. $\square$

Problem 4. Do there exist positive integers $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{101}$ such that

$$a_1 a_2 \ldots a_{101} = \sum_{1 \leqslant i < j \leqslant 101} \operatorname{lcm}(a_i, a_j),$$

where $\operatorname{lcm}(a, b)$ stands for the least common multiple of positive integers a and b?

Answer: Yes.

Solution. We will give an explicit example for which the required equality holds. Select $a_1 = a_2 = \cdots = a_{98} = 1$ and $a_{99} = 2$. Assume that $a_{100} = a > 1$ and $a_{101} = b > a$ are two coprime odd integers which we choose later. Then, $a_1 a_2 \ldots a_{101} = 2ab$. Furthermore, $\operatorname{lcm}(a_i, a_j) = 1$ for $1 \leqslant i < j \leqslant 98$ (there are $\binom{98}{2} = 49 \cdot 97$ of such pairs (i, j)), $\operatorname{lcm}(a_i, a_{99}) = 2$ for $1 \leqslant i \leqslant 98$ (there are 98 of such indices i), $\operatorname{lcm}(a_i, a_{100}) = a$ for $1 \leqslant i \leqslant 98$ (there are 98 of such indices i), $\operatorname{lcm}(a_i, a_{101}) = b$ for $1 \leqslant i \leqslant 98$ (there are 98 of such indices i), and also $\operatorname{lcm}(a_{99}, a_{100}) = 2a$, $\operatorname{lcm}(a_{99}, a_{101}) = 2b$, $\operatorname{lcm}(a_{100}, a_{101}) = ab$. Thus, equality of the problem holds if

$$2ab = 49 \cdot 97 \cdot 1 + 98 \cdot 2 + 98 \cdot a + 98 \cdot b + 2a + 2b + ab = 4949 + 100a + 100b + ab.$$

This is equivalent to

$$(a-100)(b-100) = 100^2 + 4949 = 14949 = 99 \cdot 151.$$

Select, for instance, $a = 100 + 99 = 199$ and $b = 100 + 151 = 251$. Then, the above equality clearly holds. Furthermore, since 199 is a prime number, $a = 199$ and $b = 251$ are two coprime odd integers satisfying $1 < a < b$. This gives one of possible examples. $\square$