Mathematical Competition for Students (MIFMO) of the Department of Mathematics and Informatics of Vilnius University

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(organized by Paulius Drungilas and Artūras Dubickas)

Problem 1. There are 20 clubs in a football league of some country. It turned out that after 19 rounds, when any two clubs played one game between themselves, the total number of points collected by all 20 teams is 554. (In football, after a game between two teams is finished, the winner gets 3 points, the loser gets 0 points, while both teams get 1 point each in case their game ends in a draw.)

Find the minimum number of clubs such that each of them has at least one draw.

Problem 2. Let a be a real number. For each integer $m \geqslant 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$, by the conditions

$$a_m(0) = \frac{a}{2^m} \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j) \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

Show that the limit $\lim_{n\to\infty} a_n(n+5)$ exists and find it.

Problem 3. A positive integer $n \geqslant 2$ is called a *m-powerful number* if in the prime factorization of n each prime appears with exponent at least m. Find all pairs of integers (m, k), where $m \geqslant 2$ and $k \geqslant 3$, for which there exists an increasing arithmetic progression $a_1 < a_2 < \cdots < a_k$ consisting of m-powerful numbers.

Problem 4. Find all polynomials P with real coefficients satisfying $P(0) = 1$ and $P(x)P(2x^2) = P(2x^3 + x)$ for all $x \in \mathbb{R}$.

Each problem is worth 10 points.