Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2025

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PROBLEMS

Problem 1. There are 20 clubs in a football league of some country. It turned out that after 19 rounds, when any two clubs played one game between themselves, the total number of points collected by all 20 teams is 554. (In football, after a game between two teams is finished, the winner gets 3 points, the loser gets 0 points, while both teams get 1 point each in case their game ends in a draw.)

Find the minimum number of clubs such that each of them has at least one draw.

Problem 2. Let a be a real number. For each integer $m \geqslant 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \dots$, by the conditions

$$a_m(0) = \frac{a}{2^m} \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j) \quad \text{for} \quad j = 0, 1, 2, \dots.$$

Show that the limit $\lim_{n\to\infty} a_n(n+5)$ exists and find it.

Problem 3. A positive integer $n \geqslant 2$ is called a *m-powerful number* if in the prime factorization of n each prime appears with exponent at least m. Find all pairs of integers (m, k), where $m \geqslant 2$ and $k \geqslant 3$, for which there exists an increasing arithmetic progression $a_1 < a_2 < \cdots < a_k$ consisting of m-powerful numbers.

Problem 4. Find all polynomials P with real coefficients satisfying $P(0) = 1$ and $P(x)P(2x^2) = P(2x^3 + x)$ for all $x \in \mathbb{R}$.

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. There are 20 clubs in a football league of some country. It turned out that after 19 rounds, when any two clubs played one game between themselves, the total number of points collected by all 20 teams is 554. (In football, after a game between two teams is finished, the winner gets 3 points, the loser gets 0 points, while both teams get 1 point each in case their game ends in a draw.)

Find the minimum number of clubs such that each of them has at least one draw.

Answer: 7.

Solution. Assume that there is a tournament as above with exactly k clubs having at least one draw. The total number of games is $\frac{20\cdot 19}{2} = 190$. If ℓ is the total number of draws, then the total number of points is $190 \cdot 3 - \ell = 570 - \ell = 554$, so $\ell = 16$. Since each of $20 - k$ clubs won or lost each of their games, the total number of draws is at most $\frac{k(k-1)}{2}$. This implies $16 \leqslant \frac{k(k-1)}{2}$, which is not the case for $k \leqslant 6$. Hence, $k \geqslant 7$.

Furthermore, for $k = 7$, the tournament as required exists. For example, assume that the teams A, B, C, D, E, F played all draws between themselves, and, in addition, it was a draw between A and G, but there were no draws in all other games. Then, the total number of draws is $\frac{6\cdot 5}{2} + 1 = 16$, the total number of points is 554, and exactly 7 teams, namely A, B, C, D, E, F, G, were involved in at least one draw. □

Problem 2. Let a be a real number. For each integer $m \geqslant 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$, by the conditions

$$a_m(0) = \frac{a}{2^m} \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j) \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

Show that the limit $\lim_{n\to\infty} a_n(n+5)$ exists and find it.

Answer: $e^{32a} - 1$.

Solution. Fix $n \geqslant 1$. Applying $a_n(j+1) + 1 = (a_n(j) + 1)^2$ to $j = 0, 1, \ldots, n+4$, we deduce

$$a_n(n+5) + 1 = \big(a_n(0) + 1\big)^{2^{n+5}} = \Big(1 + \frac{a}{2^n}\Big)^{2^{n+5}}.$$

By L'Hôpital's rule, we have

$$\lim_{x\to 0} \frac{\log(1+ax)}{x} = a,$$

so that

$$\lim_{x\to 0}(1+ax)^{\frac{2^5}{x}} = \lim_{x\to 0} e^{\frac{2^5 \log(1+ax)}{x}} = e^{32a}.$$

In particular, for the sequence $x = x_n = \frac{1}{2^n}$, $n = 1, 2, 3, \dots$, we find that

$$\lim_{n\to\infty} \left(1 + \frac{a}{2^n}\right)^{2^{n+5}} = e^{32a},$$

which implies $\lim_{n\to\infty} a_n(n+5) = e^{32a} - 1$. □

Problem 3. A positive integer $n \geqslant 2$ is called a *m-powerful number* if in the prime factorization of n each prime appears with exponent at least m. Find all pairs of integers (m, k), where $m \geqslant 2$ and $k \geqslant 3$, for which there exists an increasing arithmetic progression $a_1 < a_2 < \cdots < a_k$ consisting of m-powerful numbers.

Answer: All pairs of integers (m, k), where $m \geqslant 2$ and $k \geqslant 3$.

Solution 1. Fix $m \geqslant 2$. The proof is by induction on k, starting from the trivial case $k = 2$, when we can take, for instance, $a_1 = 2^m$ and $a_2 = 3^m$. Assume that for some $k \geqslant 2$ there is an increasing arithmetic progression $a_1 < a_2 < \cdots < a_k$ of m-powerful numbers with difference d. Set $a = a_k + d$. Then $a_1 < a_2 < \cdots < a_k < a$ is an arithmetic progression with a difference d. Multiplying each term of this sequence by a^m, we arrive to the $(k+1)$-term arithmetic progression

$$a_1 a^m < a_2 a^m < \cdots < a_k a^m < a^{m+1}$$

with difference da^m, whose all terms are m-powerful numbers. □

Solution 2 (by Yunus Emre Tuzcu). Take the arithmetic progression $a_j = j \cdot k!^m$ for $j = 1, \dots, k$ with difference $k!^m$. Then, for each j in the range $1 \leqslant j \leqslant k$, the integer a_j is m-powerful, because it has only prime divisors at most k, and each of those prime divisors appears with exponent at least m. □

Problem 4. Find all polynomials P with real coefficients satisfying $P(0) = 1$ and $P(x)P(2x^2) = P(2x^3 + x)$ for all $x \in \mathbb{R}$.

Answer: $P(x) = (x^2 + 1)^n$ for some integer $n \geqslant 0$.

Solution. Assume that the leading term of P is ax^m, where $a \neq 0$ and $m \geqslant 0$. For $m = 0$, by $P(0) = 1$, we get $a = 1$. Evidently, the constant polynomial $P(x) = 1$ satisfies both conditions. For $m \geqslant 1$, the leading terms of both sides are $ax^m \cdot a2^m x^{2m}$ and $a2^m x^{3m}$. This forces $a^2 = a$, and hence $a = 1$. Thus, P is monic and, as $P(0) = 1$, the product of all roots of P must be ± 1. (Clearly, zero is not a root of P.) We claim that all roots of P are unimodular. Indeed, if not, then there must be a root α of P satisfying $|\alpha| > 1$. Assume that α is the largest (in modulus) root of P. (If there are several roots of the largest moduli, then α can be any of them.) By the functional equation, $P(\alpha) = 0$ implies $P(2\alpha^3 + \alpha) = 0$, so $2\alpha^3 + \alpha$ is also a root of P. But then, by $|2\alpha^3+\alpha| \geqslant 2|\alpha|^3 - |\alpha| > |\alpha|$, we get that the root $2\alpha^3+\alpha$ of P has strictly greater modulus than that of α, a contradiction. Therefore, all roots of P are unimodular.

Now, take any root α of P. From the fact that the complex numbers $\alpha, \overline{\alpha}, 2\alpha^3+\alpha$ and its complex conjugate $2\overline{\alpha}^3+\overline{\alpha}$ are all unimodular, using $\alpha\overline{\alpha}=1$, we derive that

$$1 = (2\alpha^3+\alpha)(2\overline{\alpha}^3+\overline{\alpha}) = 4 + 2(\alpha^2+\overline{\alpha}^2) + 1 = 5 + 2\big((\alpha+\overline{\alpha})^2 - 2\big) = 1 + 2(\alpha+\overline{\alpha})^2.$$

Consequently, $\alpha+\overline{\alpha}=0$, and hence $\Re(\alpha)=0$, which is equivalent to $\{\alpha, \overline{\alpha}\} = \{i, -i\}$. Therefore, half of the roots of P are equal to i and half are equal to $-i$. It follows that m must be even and P has the form $P(x) = (x-i)^n(x+i)^n = (x^2+1)^n$ for some positive integer n. Since

$$(x^2+1)((2x^2)^2+1) = (x^2+1)(4x^4+1) = 4x^6+4x^4+x^2+1 = (2x^3+x)^2+1,$$

this polynomial clearly satisfies both conditions. □