Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2026

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PROBLEMS

Problem 1. Find the smallest positive integer k with the following property: there exist five distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$P(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients.

Problem 2. Let $\mathcal{S}$ be the set of all positive integers that can be written as

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{10}{a_{10}},$$

where $a_1, a_2, \ldots, a_{10}$ are (not necessarily distinct) positive integers.

(i) Prove that $1 \in \mathcal{S}$.
(ii) Prove that $12 \in \mathcal{S}$.
(iii) Find the largest element of $\mathcal{S}$.
(iv) Find the set $\mathcal{S}$.

Problem 3. Show that for each $a > 0$ the integral

$$\int_0^{\pi/2} \frac{(\cos x)^a}{(\sin x)^a + (\cos x)^a} dx$$

is convergent and find its value.

Problem 4. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that for each $t \in \mathbb{Z}$ the matrix $A + tB$ is invertible and that its inverse has integer entries.

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. Find the smallest positive integer k with the following property: there exist five distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$P(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients.

Answer: $k = 3$.

Solution. If $k = 1$, then $P(x)$ must be x^5, but this P does not have five distinct zeros. If $k = 2$, then $P(x) = x^5 + ax^r$ for some $a \in \mathbb{Z} \setminus \{0\}$ and $0 \leqslant r \leqslant 4$. However, such P has $x = 0$ as a double zero if $r \in \{2, 3, 4\}$, while if $r \in \{0, 1\}$ it has a nonreal zero. (The five roots of $x^5 + a$ and $(x^4 + a)x$ cannot be all real.) Therefore, such P cannot have five distinct integral zeros, and hence $k \geqslant 3$. On the other hand, the example of degree 5 polynomial

$$x(x-1)(x+1)(x-2)(x+2) = x(x^2-1)(x^2-4) = x^5 - 5x^3 + 4x$$

with five distinct integer roots and three nonzero coefficients shows that the smallest k with the required property is 3. □

Problem 2. Let $\mathcal{S}$ be the set of all positive integers that can be written as

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{10}{a_{10}},$$

where $a_1, a_2, \ldots, a_{10}$ are (not necessarily distinct) positive integers.

(i) Prove that $1 \in \mathcal{S}$.
(ii) Prove that $12 \in \mathcal{S}$.
(iii) Find the largest element of $\mathcal{S}$.
(iv) Find the set $\mathcal{S}$.

Answer: $\mathcal{S} = \{1, 2, 3, \ldots, 54, 55\}$.

Solution. We begin with the following.

Lemma 1. *For any positive integers $k \geqslant n$ there exist positive integers $a_1, a_2, \ldots, a_k$ such that*

$$n = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{k}{a_k}.$$

Proof. If $n = 1$, then selecting $a_1 = \cdots = a_k = s = 1 + 2 + \cdots + k$ we obtain

$$1 = \frac{1}{s} + \frac{2}{s} + \cdots + \frac{k}{s}.$$

If $2 \leqslant n \leqslant k$, then choosing $s = (1 + 2 + \cdots + k) - (n - 1)$ we get

$$n = \frac{1}{s} + \cdots + \frac{n-2}{s} + \frac{n-1}{1} + \frac{n}{s} + \cdots + \frac{k}{s}.$$

Thus we can select $a_j = s$ for all $j \neq n - 1$ and $a_{n-1} = 1$. This completes the proof of the lemma. □

For any positive integers $a_1, a_2, \ldots, a_{10}$ we have

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{10}{a_{10}} \leqslant \frac{1}{1} + \frac{2}{1} + \cdots + \frac{10}{1} = 1 + 2 + \cdots + 10 = 55.$$

Hence, $\mathcal{S} \subseteq \{1, 2, \ldots, 54, 55\}$. We will prove that $\mathcal{S} = \{1, 2, \ldots, 54, 55\}$. This solves part (iv) and covers all the other parts.

Fix $n \in \{1, 2, \ldots, 54, 55\}$. If $n \leqslant 10$, then $n \in \mathcal{S}$ by Lemma 1 with $k = 10$. Suppose that $11 \leqslant n \leqslant 55$. Choose the largest $k \in \{1, 2, \ldots, 9\}$ for which

$$n \leqslant k + (k + 1) + \cdots + 10.$$

Then, as $n > (k + 1) + (k + 2) + \cdots + 10$, we get $0 < n - (k + 1) - \cdots - 10 \leqslant k$. By Lemma 1, there exist positive integers $a_1, a_2, \ldots, a_k$ for which

$$n - (k + 1) - \cdots - 10 = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{k}{a_k}.$$

Now, selecting $a_{k+1} = \cdots = a_{10} = 1$, we obtain

$$n = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{k}{a_k} + \frac{k+1}{1} + \cdots + \frac{10}{1},$$

and hence $n \in \mathcal{S}$. □

Problem 3. Show that for each $a > 0$ the integral

$$\int_0^{\pi/2} \frac{(\cos x)^a}{(\sin x)^a + (\cos x)^a} dx$$

is convergent and find its value.

Answer: $\frac{\pi}{4}$.

Solution. Set

$$I(a) = \int_0^{\pi/2} \frac{(\cos x)^a}{(\sin x)^a + (\cos x)^a} dx = \int_0^{\pi/2} \frac{1}{1 + (\tan x)^a} dx.$$

Substituting $y = \tan x$ we see that $dy = \frac{dx}{(\cos x)^2} = (1+y^2)dx$, so

$$I(a) = \int_0^\infty \frac{1}{(1+y^a)(1+y^2)} dy.$$

Since the integrand satisfies

$$0 < \frac{1}{(1+y^a)(1+y^2)} < \frac{1}{1+y^2}$$

and $\int_0^\infty \frac{dy}{1+y^2} = \pi/2$, the integral $I(a)$ is convergent for every $a > 0$. (Alternatively, one can verify that at the endpoints $0, \pi/2$ of the interval $[0, \pi/2]$ the integrand $\frac{(\cos x)^a}{(\sin x)^a + (\cos x)^a}$ is 1 and 0 respectively. Since for any $a > 0$ the function is continuous and bounded on the interval $[0, \pi/2]$, the integral is a proper Riemann integral and is therefore convergent.)

Substituting x by $\pi/2 - x$ in $I(a)$ we obtain

$$I(a) = \int_0^{\pi/2} \frac{(\cos x)^a}{(\sin x)^a + (\cos x)^a} dx = \int_0^{\pi/2} \frac{(\sin x)^a}{(\sin x)^a + (\cos x)^a} dx.$$

Therefore,

$$2I(a) = \int_0^{\pi/2} \frac{(\cos x)^a + (\sin x)^a}{(\sin x)^a + (\cos x)^a} dx = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2},$$

which implies $I(a) = \pi/4$. □

Problem 4. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that for each $t \in \mathbb{Z}$ the matrix $A + tB$ is invertible and that its inverse has integer entries.

Solution. We first claim that a square matrix M with integer entries has an inverse with integer entries if and only if $\det M = \pm 1$. Indeed, if N is the inverse of M, then

$$\det M \cdot \det N = \det(MN) = 1,$$

so $\det M = \pm 1$. Conversely, if $\det M = \pm 1$, then $\pm M'$ is an inverse with integer entries, where M' is the classical adjoint of M. This completes the proof of the claim. (Alternatively, for 2×2 integer matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $\delta = ad - bc = \pm 1$ observe that the integer matrix $\begin{pmatrix} \delta d & -\delta b \\ -\delta c & \delta a \end{pmatrix}$ is the inverse of M.)

Set $f(x) = \det(A + xB)$. Then $f(x)$ is a polynomial in $\mathbb{Z}[x]$ of degree at most 2 such that $f(x) \in \{-1, 1\}$ for $x = 0, 1, 2, 3, 4$. Thus, by the Pigeonhole Principle, f takes some value $\theta \in \{-1, 1\}$ at three or more distinct points x. But the only polynomials of degree at most 2 that take the same value θ three or more times are the constant polynomials

$f(x) = \theta$. This implies that $\det(A + xB) = \theta$ for each $x \in \mathbb{R}$. In particular, inserting $x = t \in \mathbb{Z}$, we obtain

$$\det(A + tB) = \theta = \pm 1.$$

Hence, by the above claim, the matrix $A + tB$ with integer entries has an inverse with integer entries. $\square$